Lecture 14 03/04/2019

Maxwell Equations and Electrodynamics (Cont'd)

Conservation of Angular momentum

From the Conservation of momentum, discussed last time, we have:

$$\int_V \left[\frac{\partial}{\partial t}\vec{p}_{mech} + \frac{\partial}{\partial t}\frac{1}{c}(\vec{E}\times\vec{H})\right] d^3r = \hat{e}_i \int_V \partial_j T_{ij}\, d^3r \Rightarrow$$

$$\int_V \left[\frac{\partial}{\partial t}(\vec{x}\times\vec{p}_{mech}) + \frac{\partial}{\partial t}\left(\vec{x}\times\frac{1}{c}(\vec{E}\times\vec{H})\right)\right] d^3r = \int_V (\vec{x}\times\hat{e}_i\, \partial_j T_{ij})\, d^3r$$

Note that:

$$\vec{x}\times(\hat{e}_i\,\partial_j T_{ij}) = \partial_j(\vec{x}\times\hat{e}_i\, T_{ij}) - \underbrace{(\partial_j\vec{x})}_{\hat{e}_j}\times\hat{e}_i\, T_{ij}$$

Therefore, we can write: ($\hat{e}_i\times\hat{e}_j$ anti-symmetric, T_{ij} symmetric)

$$\frac{d}{dt}(\vec{L}_{mech}+\vec{J}_{EM}) = \oint_S \vec{x}\times\hat{e}_i\, T_{ij}\, n_j\, da + \int \cancel{(\hat{e}_i\times\hat{e}_j)}\, T_{ij}\, d^3r$$

$$\boxed{\Rightarrow \frac{d}{dt}(\vec{L}_{mech}+\vec{J}_{EM}) = \oint_S \vec{x}\times\hat{e}_i\, T_{ij}\, n_j\, da}$$

Here:

$$\vec{L}_{mech} = \int_{V} (\vec{X} \times \vec{P}_{mech})\, d^3r \quad , \quad \vec{J}_{EM} = \int_{V} \vec{X} \times \frac{1}{c^2} (\vec{E} \times \vec{H})\, d^3r$$

$\vec{J}_{EM}$ is the angular momentum of the electromagnetic field.

After using $\vec{B} = \vec{\nabla} \times \vec{A}$, we have:

$$\vec{J}_{EM} = \varepsilon_0 \int_{V} \vec{X} \times (\vec{E} \times (\vec{\nabla} \times \vec{A}))\, d^3r = \varepsilon_0 \int \left[\vec{X} \times (E_k \vec{\nabla} A_k) - \vec{X} \times (E_k \partial_k \vec{A})\right] d^3r$$

$$= \varepsilon_0 \int_{V} \left[E_k\, i \vec{L}\, A_k - E_k\, \vec{X} \times \partial_k \vec{A} \right] d^3r$$

Here $\vec{L} \equiv -i\, \vec{X} \times \vec{\nabla}$ is the orbital angular momentum operator. We note that:

$$\vec{X} \times (\partial_k \vec{A}) = \partial_k (\vec{X} \times \vec{A}) - \underbrace{(\partial_k \vec{X})}_{\hat{e}_k} \times \vec{A}$$

Thus:

$$\vec{J}_{EM} = \varepsilon_0 \int_{V} \left[i E_k \vec{L} A_k + E_k \hat{e}_k \times \vec{A} - E_k \partial_k (\vec{X} \times \vec{A}) \right] d^3r$$

But:

$$E_k \partial_k (\vec{X} \times \vec{A}) = \partial_k \left(E_k (\vec{X} \times \vec{A})\right) - \underbrace{(\partial_k E_k)}_{\vec{\nabla} \cdot \vec{E} = 0 \text{ (no charge)}} (\vec{X} \times \vec{A}) = \partial_k (E_k (\vec{X} \times \vec{A})) \Rightarrow$$

$$\int_{V} E_k \partial_k (\vec{X} \times \vec{A})\, d^3r = \int_{V} \partial_k (E_k (\vec{X} \times \vec{A}))\, d^3r = \oint_S E_k (\vec{X} \times \vec{A})\, n_k\, da$$

For localized fields, the surface integral vanishes when the boundary S is chosen at infinity. Hence:

$$\vec{J}_{EM} = \varepsilon_0 \int_V i E_k \vec{L} A_k \, d^3 r + \varepsilon_0 \int_V (\vec{E} \times \vec{A}) \, d^3 r$$

The first term on the right-hand side is the orbital part of the angular momentum of the electromagnetic field, while the second term can be identified with the spin part. It is important to note that the above expression is not valid for plane waves as they are not localized in space.

Complex Notation

If all sources of fields are sinusoidal functions of time, then we may write for the radiated/scattered fields (which are also sinusoidal):

$$\vec{E}(\vec{x}, t) = \mathrm{Re}\left[\vec{E}(\vec{x}) \, e^{-i\omega t}\right] = |\vec{E}(\vec{x})| \cos(\omega t - \phi_E(\vec{x}))$$

absolute value in the sense of complex numbers)

$$\vec{H}(\vec{x}, t) = \mathrm{Re}\left[\vec{H}(\vec{x}) \, e^{-i\omega t}\right] = |\vec{H}(\vec{x})| \cos(\omega t - \phi_H(\vec{x}))$$

The Maxwell equations can be written in term of the complex electric and magnetic fields:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = i\omega \vec{B} \quad , \quad \vec{\nabla} \times \vec{H} = \vec{J} + \varepsilon \frac{\partial \vec{E}}{\partial t} = \vec{J} - i\omega\varepsilon \vec{E}$$

For quantities that are bilinear in electric and magnetic fields, like the Poynting vector or the momentum/angular momentum density, we have to be careful. For example, consider the Poynting vector $\vec{S} = \vec{E} \times \vec{H}$. We cannot write a complex Poynting vector that is just the cross product of complex $\vec{E}$ and $\vec{H}$ fields and then take its real part. The reason is that $Re(\vec{E} \times \vec{H}) \neq Re(\vec{E}) \times Re(\vec{H})$.

However, we can write $\vec{S}$ in complex notation in a way that it is real:

$$\vec{S} = \frac{1}{2}\left[\vec{E}(\vec{x})\, e^{-i\omega t} + \vec{E}^*(\vec{x})\, e^{i\omega t}\right] \times \frac{1}{2}\left[\vec{H}(\vec{x})\, e^{-i\omega t} + \vec{H}^*(\vec{x})\, e^{i\omega t}\right] =$$

$$\frac{1}{4}\left[\vec{E}(\vec{x}) \times \vec{H}(\vec{x})\, e^{-2i\omega t} + c.c.\right] + \frac{1}{4}\left[\vec{E}(\vec{x}) \times \vec{H}^*(\vec{x}) + \vec{E}^*(\vec{x}) \times \vec{H}(\vec{x})\right]$$

It is seen that:

$$\langle \vec{S} \rangle_{time} = \frac{1}{4}\left(\vec{E} \times \vec{H}^* + \vec{E}^* \times \vec{H}\right) = Re\left(\frac{1}{2}\vec{E} \times \vec{H}^*\right)$$

Then, the Poynting theorem for complex harmonic fields reads:

$$-Re\left(\oint_S \frac{1}{2}(\vec{E} \times \vec{H}^*) \cdot \hat{n}\, da\right) = Re\left(\frac{1}{2}\int_V \vec{E} \cdot \vec{J}^*\, d^3x\right) + \frac{1}{2} Re\left(\int_V i\omega(\epsilon^* |\vec{E}|^2 - \mu |\vec{H}|^2)\, d^3x\right)$$

The term on the left-hand side is the time-averaged rate of the energy flow into the volume V. The first term on the right-hand side represents the time-averaged rate of dissipation of energy due to the work done on the current distribution $\vec{J}$. The second term on the right-hand side represents the work done by the electromagnetic field on the bound charges and currents (hence zero in free space). Note that this term vanishes if ϵ and μ are real.